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Pair Production in a Uniform Electric Field

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We study the Klein–Gordon field coupled with an external uniform vector potential. We compute pair production in a finite time t using the semiclassical approximation, and show that, after the interaction of the Klein–Gordon field with the external potential, when $\hbar \to 0$ the average number of produced pairs is zero. There is agreement with the classical limit because the classical limit involves no production of pairs. We compared our results with those of Schwinger. Finally we saw that the random variable $N(t)=''$ net number of pairs produced at time t'' is in the semiclassical limit a stochastic Poisson process.

KEY WORDS: pair production; Schwinger's formula; semiclassical approach.

1. INTRODUCTION

The subject of this paper is the study of pair production at each finite time t due to the presence of an external uniform vector potential $\frac{d}{dt}\vec{f}(t) \in \mathcal{C}_0^{\infty}(0, \infty)$. The adiabatic approach for large time was studied in many works (Berger, 1975; Birrell and Davies, 1984; Eisenberg and Kälbermann, 1988; Fulling, 1985). For this reason we are interested in the pair production at finite time.

The pair production in a finite time was studied in Parker (1969) using the Heisenberg picture, in the context of the expanding universes. In this work we follow an analogous formalism developed in Parker (1969), but we use the Schrödinger picture (like (Berger, 1975)), because is more easy to calculate a semiclassical solution of the second quantized Klein–Gordon field equation.

In the first section we develop the diagonalization method for the second quantized Klein–Gordon field, defined at $[-L, L]^3$ with periodic boundary conditions (like (Berger, 1975; Greiner *et al.*, 1985)). First we will see that the Klein–Gordon equation is equivalent to a Hamiltonian system, composed of an infinite number of harmonic oscillators with frequencies which depend on time. Once we have seen this equivalence, we quantize these oscillators and obtain the time dependent energy and the electric charge operators. From the energy operator, we

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obtain the quantum equation of the Klein–Gordon field, i.e., the second quantized Klein–Gordon field equation. We also see that we can find all the eigenfunctions of the energy and the electric charge operators. We observe that these eigenfunctions clearly depend on time. Consequently, the vacuum state, the state of a particle, the state of an antiparticle, etc. depend on time. This is a consequence of the electric field $\vec{E}(t) = \frac{1}{c}\frac{d}{dt}\vec{f}(t)$, produced by the external potential $\vec{f}(t)$. Finally, with all these eigenfunctions, we can construct the Fock space.

We then study the semiclassical dynamics of the vacuum state, using the semiclassical solution, for the initial vacuum state, of the second quantized Klein–Gordon field equation, and we calculate the probability that the vacuum state remains unchanged in the semiclassical approximation, i.e., the semiclassical probability that pairs are not produced at finite time t.

If we denote this probability by $P_\hbar(t)$, we show that

$$P_\hbar(t) \sim \exp\left(-\frac{\alpha}{64}\frac{\varepsilon(t)}{mc^2}\right), \tag{1}$$

where $\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant, $\varepsilon(t) = \frac{(2L)^3}{8\pi}|\vec{E}(t)|^2$ is the energy of the external field at time t, and the symbol "$\sim$" means approximately in the sense that, $a \sim b$ if $\lim_{\hbar\to 0}(a-b) = 0$, for fixed α.

In Section 3 we show that, if we calculate the probability that pairs ara not produced at time t, using Born's approximation to the solution of the second quantized Klein–Gordon field equation, we obtain the formula (1).

It is important to remark that for larger times, i.e., when the electric field is zero, formula (1) becomes

$$P_\hbar(t) = \exp(O(\hbar^\infty)),$$

this result is explained in more detail in the Appendix A. In general, we do not have an explicit expression of the formula (1), for large times. For this reason, in Section 4 we study the particular case $\vec{f}(t) = (0, 0, \chi(t))$, where

$$\chi(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ cEt & \text{if} \quad 0 < t < T \\ cEt & \text{if} \quad t > T. \end{cases}$$

For this potential, when $t > T$, using the WKB approximation in the complex plane, we obtain the following explicit expression of the formula (1)

$$\exp\left(-\frac{TL^3E^2\alpha}{\pi^3\hbar}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\exp\left(-\frac{n\pi m^2c^4}{\hbar c|eE|}\right)\right), \tag{2}$$

i.e., we obtain the Schwinger's formula (Greiner *et al.*, 1985; Holstein, 1999; Itzykson and Znber, 1980; Nikishov, 1970; Popov, 1972; Schwinger, 1951).

Note that, we obtain Schwinger's formula in the adiabatic approach, because the method of imaginary times (Marinov and Popov, 1977; Popov, 1972), i.e., the WKB approximation in the complex plane, used in the computation of the Schwinger's formula is only justified in the adiabatic approach (see (Bonet *et al.*, 1998; Fedoryuk, 1993; Meyer, 1980; Wasow, 1973)).

In Section 5, we will see that the pair production is, in the semiclassical approximation, a stochastic Poisson process, in agreement with the work (Schiff, 1968), where Schwinger tried to show that pair production in presence of an external field is a Poisson process.

Finally, in the mathematical Appendix A, we give the semiclassical solution, for the vacuum state, of the second quantized Klein–Gordon field equation. It is important to remark that it is not possible to apply the WKB approximation in this problem, however it is possible to apply a generalization of the WKB method, called Maslov method (like (Haro, 1998; Maslov and Fedoriuk, 1981)), but we belive that this method is excessively complex. For this reason, we use a more easy method explained in detail in the Appendix A.

2. THE SECOND QUANTIZED KLEIN–GORDON FIELD COUPLED WITH A UNIFORM EXTERNAL VECTOR POTENTIAL

In this section we diagonalize the Hamiltonian following an analogous treatment used in Berger (1975).

The Lagrangian and the energy of the Klein–Gordon field at time t, in the domain $[-L, L]^3$, with periodic boundary conditions are (Greiner *et al.*, 1985),

$$L(t) = \int_{[-L,L]^3} \left(\hbar^2 |\partial_t \psi|^2 - c^2 \left| \left(-i\hbar\vec{\nabla} + \frac{e}{c}\vec{f}(t) \right) \psi \right|^2 - m^2c^4|\psi|^2 \right) d\vec{x}$$

$$E(t) = \int_{[-L,L]^3} \left(\hbar^2 |\partial_t \psi|^2 + c^2 \left| \left(-i\hbar\vec{\nabla} + \frac{e}{c}\vec{f}(t) \right) \psi \right|^2 + m^2c^4|\psi|^2 \right) d\vec{x}.$$

The electric charge is

$$\rho(t) = i\hbar \int_{[-L,L]^3} \left(\partial_t\psi\,\psi^* - \psi\,\partial_t\psi^* \right) d\vec{x}.$$

If we expand ψ in Fourier's series, $\psi(\vec{x}, t) = \sum_{\vec{k}\in\mathbb{Z}^3} A_{\vec{k}}(\vec{x})$, where we have

$$\psi_{\vec{k}}(\vec{x}) = \exp\left(\frac{i\pi}{L}\vec{k}\cdot\vec{x} \right) \Big/ (2L)^{\frac{3}{2}},$$

then

$$L(t) = \sum_{\vec{k}\in\mathbb{Z}^3} \hbar^2 |\dot{A}_{\vec{k}}|^2 - \epsilon_{\vec{k}}^2(t)|A_{\vec{k}}|^2,$$

where

$$\epsilon_{\vec{k}}(t) = \sqrt{c^2 \left| \frac{\pi \hbar \vec{k}}{L} + \frac{e}{c} \vec{f}(t) \right|^2 + m^2 c^4}.$$

Using the momenta $B_{\vec{k}} = \hbar^2 \dot{A}_{\vec{k}}$, we obtain

$$E(t) = \sum_{\vec{k}\in\mathbb{Z}^3} \frac{|B_{\vec{k}}|^2}{\hbar^2} + \epsilon_{\vec{k}}^2(t)|A_{\vec{k}}|^2; \qquad \rho(t) = \sum_{\vec{k}\in\mathbb{Z}^3} \frac{i}{\hbar}\left(A_{\vec{k}}^* B_{\vec{k}} - A_{\vec{k}} B_{\vec{k}}^*\right).$$

If we make the real canonical change

$$B_{\vec{k}} = \frac{\hbar}{\sqrt{2}}(P_{\vec{k}} + i\,\bar{P}_{\vec{k}}); \qquad A_{\vec{k}} = \frac{1}{\hbar\sqrt{2}}(Q_{\vec{k}} + i\,\bar{Q}_{\vec{k}}),$$

and let $\omega_{\vec{k}}(t) = \frac{\epsilon_{\vec{k}}(t)}{\hbar}$ be the corresponding frequency, then $E(t)$ and $\rho(t)$ take the form

$$E(t) = \frac{1}{2}\sum_{\vec{k}\in\mathbb{Z}^3} \left(P_{\vec{k}}^2 + \omega_{\vec{k}}^2(t) Q_{\vec{k}}^2\right) + \left(\bar{P}_{\vec{k}}^2 + \omega_{\vec{k}}^2(t) \bar{Q}_{\vec{k}}^2\right)$$

$$\rho(t) = \frac{1}{\hbar}\sum_{\vec{k}\in\mathbb{Z}^3} (\bar{Q}_{\vec{k}} P_{\vec{k}} - Q_{\vec{k}} \bar{P}_{\vec{k}}).$$

This is the decomposition of the energy into oscillators. Notice that the Klein–Gordon equation is equivalent to the Hamiltonian system

$$\begin{cases} \dot{Q}_{\vec{k}} = P_{\vec{k}} \\ \dot{P}_{\vec{k}} = -\omega_{\vec{k}}^2(t) Q_{\vec{k}} \end{cases} \qquad \begin{cases} \dot{\bar{Q}}_{\vec{k}} = \bar{P}_{\vec{k}} \\ \dot{\bar{P}}_{\vec{k}} = -\omega_{\vec{k}}^2(t) Q_{\vec{k}} \end{cases} \tag{3}$$

To obtain the quantum theory we now quantize these oscillators, i.e., $P_{\vec{k}} \to -i\hbar\partial_{Q_{\vec{k}}}$, $\bar{P}_{\vec{k}} \to -i\hbar\partial_{\bar{Q}_{\vec{k}}}$, and the equation becomes

$$i\hbar\partial_t\Phi = \frac{1}{2}\sum_{\vec{k}\in\mathbb{Z}^3} \left[\left(-\hbar^2\partial_{Q_{\vec{k}}}^2 + \omega_{\vec{k}}^2(t) Q_{\vec{k}}^2\right) + \left(-\hbar^2\partial_{\bar{Q}_{\vec{k}}}^2 + \omega_{\vec{k}}^2(t)\bar{Q}_{\vec{k}}^2\right)\right]\Phi - \sum_{\vec{k}\in Z^n} \epsilon_{\vec{k}}(t)\Phi.$$

Now we look for the eigenfunctions of the energy and of the electric charge operators. First, we have to introduce the creation and annihilation operators for particles and antiparticles, at time t.

$$\hat{a}_{\vec{k}}(t) = \frac{1}{2\sqrt{\epsilon_{\vec{k}}(t)}}\left[\left(\hbar\partial_{Q_{\vec{k}}} + \omega_{\vec{k}}(t) Q_{\vec{k}}\right) + i\left(\hbar\partial_{\bar{Q}_{\vec{k}}} + \omega_{\vec{k}}(t)\bar{Q}_{\vec{k}}\right)\right]$$

$$\hat{a}_{\vec{k}}^{+}(t) = \frac{1}{2\sqrt{\epsilon_{\vec{k}}(t)}}\left[\left(-\hbar\partial_{Q_{\vec{k}}} + \omega_{\vec{k}}(t) Q_{\vec{k}}\right) - i\left(-\hbar\partial_{\bar{Q}_{\vec{k}}} + \omega_{\vec{k}}(t)\bar{k}\right)\right]$$

$$\hat{b}_{-\vec{k}}(t) = \frac{1}{2\sqrt{\epsilon_{\vec{k}}(t)}}\left[\left(\hbar\partial_{Q_{\vec{k}}} + \omega_{\vec{k}}(t)Q_{\vec{k}}\right) - i\left(\hbar\partial_{\bar{Q}_{\vec{k}}} + \omega_{\vec{k}}(t)\bar{Q}_{\vec{k}}\right)\right]$$

$$\hat{b}^{+}_{-\vec{k}}(t) = \frac{1}{2\sqrt{\epsilon_{\vec{k}}(t)}}\left[\left(-\hbar\partial_{Q_{\vec{k}}} + \omega_{\vec{k}}(t)Q_{\vec{k}}\right) + i\left(-\hbar\partial_{\bar{Q}_{\vec{k}}} + \omega_{\vec{k}}(t)\bar{k}\right)\right].$$

Then

$$\hat{E}(t) = \sum_{\vec{k}\in\mathbb{Z}^3} \epsilon_{\vec{k}}(t)\left(a^{+}_{\vec{k}}(t)a_{\vec{k}}(t) + b^{+}_{-\vec{k}}(t)_{-\vec{k}}(t)\right)$$

$$\hat{\rho}(t) = \sum_{\vec{k}\in\mathbb{Z}^3} \left(a^{+}_{\vec{k}}(t)a_{\vec{k}} - b^{+}_{-\vec{k}}(t)b_{-\vec{k}}(t)\right).$$

Now, we construct the vacuum state at time t (Berger, 1975; Greiner *et al.*, 1985; Haro, 1997).

If we consider

$$\phi^{0,0}_{\vec{k}}(Q_{\vec{k}}, \bar{Q}_{\vec{k}}, t) = \sqrt{\frac{\omega_{\vec{k}}(t)}{\pi\hbar}}\exp\left(-\frac{\omega_{\vec{k}}(t)}{2\hbar}\left(Q^2_{\vec{k}} + \bar{Q}^2_{\vec{k}}\right)\right),$$

then the vacuum state at time t, $|0\rangle(t)$, is

$$|0\rangle(t) = \prod_{\vec{k}\in\mathbb{Z}^3} \phi^{0,0}_{\vec{k}}(Q_{\vec{k}}, \bar{Q}_{\vec{k}}, t), \tag{4}$$

since

$$\hat{E}(t)|0\rangle(t) = 0 \qquad \hat{\rho}(t)|0\rangle(t) = 0.$$

Starting from this state we can define another set of states (Berger, 1975; Bjorken and Drell, 1965; Greiner *et al.*, 1985). In fact, we consider series

$$\{n_{\vec{k}}\} : \begin{array}{l} \mathbb{Z}^n \to \mathbb{N} \\ \vec{k} \to n_{\vec{k}} \end{array}$$

and let

$$|\{n_{\vec{k}}\};\{m_{\vec{k}}\}\rangle(t) \prod_{\vec{k}\in\mathbb{Z}^3} \frac{\left(\hat{a}^{+}_{\vec{k}}(t)\right)^{n_{\vec{k}}}}{\sqrt{n_{\vec{k}}!}} \frac{\left(\hat{b}^{+}_{-\vec{k}}(t)\right)^{m_{\vec{k}}}}{\sqrt{m_{\vec{k}}!}}|0\rangle(t).$$

Then $|\{n_{\vec{k}}\};\{m_{\vec{k}}\}\rangle(t)$, satisfies

$$\hat{E}(t)|\{n_{\vec{k}}\};\{m_{\vec{k}}\}\rangle(t) = \sum_{\vec{l}\in\mathbb{Z}^3} \epsilon_{\vec{l}}(t)(n_{\vec{l}} + m_{\vec{l}})|\{n_{\vec{k}}\};\{m_{\vec{k}}\}\rangle(t)$$

$$\hat{\rho}(t)|\{n_{\vec{k}}\};\{m_{\vec{k}}\}\rangle(t) = \sum_{\vec{l}\in\mathbb{Z}^3}(n_{\vec{l}} - m_{\vec{l}})\{n_{\vec{k}}\};\{m_{\vec{k}}\}\rangle(t).$$

Therefore, at time t the state $|\{n_{\vec{k}}\};\{m_{\vec{k}}\}\rangle(t)$ contains $n_{\vec{k}}$ particles and $m_{\vec{k}}$ antiparticles with energy $\epsilon_{\vec{k}}(t)$ for each $\vec{k} \in \mathbb{Z}^3$.

3. VACUUM TO VACUUM TRANSITIONS

Here we study the dynamics of the vacuum. Let $\mathcal{T}_\hbar^t|0\rangle(0)$ be the solution to the problem

$$\begin{cases} i\hbar\partial_t|\Psi\rangle = \hat{E}(t)|\Psi\rangle \\ |\Psi\rangle(0) = |0\rangle(0). \end{cases} \tag{5}$$

Then $\mathcal{T}_\hbar^t|0\rangle(0) = \prod_{\vec{k}\in Z^n} T_\hbar^t \phi_\hbar^{0,0}(Q_{\vec{k}}, \bar{Q}_{\vec{k}}, 0)$, where $T_\hbar^t \phi_{\vec{k}}^{0,0}(Q_{\vec{k}}, \bar{Q}_{\vec{k}}, 0)$ is the solution of the problem

$$\begin{cases} i\hbar\partial_t\phi = \left[\frac{1}{2}\left(-\hbar^2\partial^2_{Q_{\vec{k}}} + \omega^2_{\vec{k}}(t)Q^2_{\vec{k}} - \hbar^2\partial^2_{\bar{Q}_{\vec{k}}} + \omega^2_{\vec{k}}(t)\bar{Q}^2_{\vec{k}}\right) - \epsilon_{\vec{k}}(t)\right]\phi \\ \phi(0) = \phi_{\vec{k}}^{0,0}(Q_{\vec{k}}, \bar{Q}_{\vec{k}}, 0). \end{cases} \tag{6}$$

Let $P_\hbar(t) = |(t)\langle 0|\mathcal{T}_\hbar^t 0\rangle(0)|^2$ represent the probability that pairs are not produced at time t.

Then, we have the following:

Theorem 3.1. *If we suppose that* $\vec{f} \in \mathcal{C}_0^\infty(0, \infty)$, *then we have*

$$P_\hbar(t) \sim \exp\left(-\frac{\alpha}{64}\frac{\varepsilon(t)}{mc^2}\right).$$

We prove Theorem 3.1 in Appendix A. Here we deduce the result of the theorem using Born's approximation.

We introduce the free creation and annihilation operators

$$\begin{aligned}
\hat{a}_{\vec{k}} &= \frac{1}{2\sqrt{\epsilon_{\vec{k}}}}\left[\left(\hbar\partial_{Q_{\vec{k}}} + \omega_{\vec{k}}Q_{\vec{k}}\right) + i\left(\hbar\partial_{\bar{Q}_{\vec{k}}} + \omega_{\vec{k}}\bar{Q}_{\vec{k}}\right)\right] \\
\hat{a}^+_{\vec{k}} &= \frac{1}{2\sqrt{\epsilon_{\vec{k}}}}\left[\left(-\hbar\partial_{Q_{\vec{k}}} + \omega_{\vec{k}}Q_{\vec{k}}\right) - i\left(-\hbar\partial_{\bar{Q}_{\vec{k}}} + \omega_{\vec{k}}\bar{Q}_{\vec{k}}\right)\right] \\
\hat{b}_{-\vec{k}} &= \frac{1}{2\sqrt{\epsilon_{\vec{k}}}}\left[\left(\hbar\partial_{Q_{\vec{k}}} + \omega_{\vec{k}}Q_{\vec{k}}\right) - i\left(\hbar\partial_{\bar{Q}_{\vec{k}}} + \omega_{\vec{k}}\bar{Q}_{\vec{k}}\right)\right] \\
\hat{b}^+_{-\vec{k}} &= \frac{1}{2\sqrt{\epsilon_{\vec{k}}}}\left[\left(-\hbar\partial_{Q_{\vec{k}}} + \omega_{\vec{k}}Q_{\vec{k}}\right) + i\left(-\hbar\partial_{\bar{Q}_{\vec{k}}} + \omega_{\vec{k}}\bar{Q}_{\vec{k}}\right)\right],
\end{aligned}$$

where

$$\omega_{\vec{k}} \equiv \frac{\epsilon_{\vec{k}}}{\hbar} = \frac{1}{\hbar}\sqrt{\frac{c^2\pi^2\hbar^2|\vec{k}|^2}{L^2} + m^2c^4}.$$

We also introduce the operators $\hat{\gamma}_{\vec{k}} = \hat{a}_{\vec{k}} + \hat{b}^+_{-\vec{k}}$, $\hat{\gamma}^+_{\vec{k}} = \hat{a}^+_{\vec{k}} + \hat{b}_{-\vec{k}}$.

Then, the quantum Hamiltonian operator is

$$\hat{H}(t) = \hat{H}_0 + \sum_{\vec{k}\in\mathbb{Z}^3} \frac{G_{\vec{k}}(t)}{2\epsilon_{\vec{k}}} : \hat{\gamma}_{\vec{k}}^{+} \hat{\gamma}_{\vec{k}} :, \tag{7}$$

where $G_{\vec{k}}(t) = \epsilon_{\vec{k}}^2(t) - \epsilon_{\vec{k}}^2$, $\hat{H}_0 \equiv \sum_{\vec{k}\in\mathbb{Z}^3} \hat{H}_{\vec{k},0} = \sum_{\vec{k}\in\mathbb{Z}^3} \epsilon_{\vec{k}}(\hat{a}_{\vec{k}}^{+}\hat{a}_{\vec{k}} + \hat{b}_{-\vec{k}}^{+})$, is the free quantum Hamiltonian operator, and :: is the normal ordering operator.

Now, we study the problem

$$\begin{cases} i\hbar\partial_t \phi = \hat{H}_{\vec{k}}(t)\phi \\ \phi(0) = \phi_{\vec{k}}^{0,0}, \end{cases} \tag{8}$$

with $\hat{H}_{\vec{k}}(t) = \hat{H}_{\vec{k},0} + \frac{G_{\vec{k}}(t)}{2\epsilon_{\vec{\hbar}}} : \hat{\gamma}_{\vec{k}}^{+}\hat{\gamma}_{\vec{k}}:$ and $\phi_{\vec{k}}^{0,0}(Q_{\vec{k}}, \bar{Q}_{\vec{k}}) = \sqrt{\frac{\omega_{\vec{k}}}{\pi\hbar}} \exp(-\frac{\omega_{\vec{k}}}{2\hbar}(Q_{\vec{k}}^2 + \bar{Q}_{\vec{k}}^2))$.

For a fixed t, using the perturbation theory, we obtain the following eigenfunctions for the operator $\hat{H}_{\vec{k}}(t)$:

$$\bar{\phi}_{\vec{k}}^{0,0}(t) \sim \phi_{\vec{k}}^{0,0} - \frac{G_{\vec{k}}(t)}{4\epsilon_{\vec{k}}^2}\phi_{\vec{k}}^{1,1}$$

$$\bar{\phi}_{\vec{k}}^{1,1}(t) \sim \phi_{\vec{k}}^{1,1} - \frac{G_{\vec{k}}(t)}{4\epsilon_{\vec{k}}^2}\left(\phi_{\vec{k}}^{2,2} - \phi_{\vec{k}}^{0,0}\right)$$

etc., where $\phi_{\vec{k}}^{s,s} = \frac{(\hat{a}_{\vec{\hbar}}^{+})^s(\hat{b}_{-\vec{k}}^{+})}{s!}\phi_{\vec{k}}^{0,0}$ with $s \in \mathbb{N}$.

In Born's approximation, the solution to problem (8) is

$$\mathcal{T}_{\hbar}^{t}\phi_{\vec{k}}^{0,0} \sim \phi_{\vec{k}}^{0,0} - \frac{i}{\hbar}\int_0^t \frac{G_{\vec{k}}(t)}{2\epsilon_{\vec{k}}} \exp\left(-\frac{2i}{\hbar}(t-\tau)\epsilon_{\vec{k}}\right) d\tau \phi_{\vec{k}}^{1,1}$$

$$\sim \phi_{\vec{k}}^{0,0} - \frac{G_{\vec{k}}(t)}{4\epsilon_{\vec{k}}^2}\phi_{\vec{k}}^{1,1} - \frac{i\hbar\dot{G}_{\vec{k}}(t)}{8\epsilon_{\vec{k}}^3}\phi_{\vec{k}}^{1,1}.$$

Then, the probability that a pair is created at time t is (see (Landau and Lifchitz, 1967, p. 172))

$$\left|\int_{\mathbb{R}^2} \bar{\phi}_{\vec{k}}^{1,1}(t)\mathcal{T}_{\hbar}^{t}\phi_{\vec{k}}^{0,0}\, dQ_{\vec{k}}\, d\bar{Q}_{\vec{k}}\right|^2 \sim \frac{\hbar^2|\dot{G}_{\vec{k}}(t)|^2}{64\epsilon_{\vec{k}}^6} = \frac{\hbar^3}{16\epsilon_{\vec{k}}^6}\left[\frac{\alpha c^5\pi^2\hbar^2}{L^2}(\vec{E}(t)\cdot\vec{k})^2 \right.$$

$$\left. + \frac{2c^3e^3\pi}{L}(\vec{E}(t)\cdot\vec{k})(\vec{E}(t)\cdot\vec{f}(t)) + \hbar\alpha^2c^4(\vec{E}(t)\cdot\vec{f}(t))^2\right]$$

From this we can deduce that,

$$P_\hbar(t) \sim \prod_{\vec{k}\in\mathbb{Z}^3}\left(1 - \frac{\hbar^3}{16\epsilon_{\vec{k}}^6}\left[\frac{\alpha c^5\pi^2\hbar^2}{L^2}(\vec{E}(t)\cdot\vec{k})^2 + \frac{2c^3e^3\pi}{L}(\vec{E}(t)\cdot\vec{k})(\vec{E}(t)\cdot\vec{f}(t)) + \hbar\alpha^2c^4(\vec{E}(t)\cdot\vec{f}(t))^2\right]\right)$$
$$\sim \exp\left(-\sum_{\vec{k}\in\mathbb{Z}^3}\frac{\hbar^3}{16\epsilon_{\vec{k}}^6}\left[\frac{\alpha c^5\pi^2\hbar^2}{L^2}(\vec{E}(t)\cdot\vec{k})^2 + \hbar\alpha^2c^4(\vec{E}(t)\cdot\vec{f}(t))^2\right]\right),$$

Now, since $\hbar$ is small, using the Riemann's integral definition, we have approximately

$$P_\hbar(t) \sim \exp\left(-\frac{L^3}{\pi^3}\int_{\mathbb{R}^3}\frac{d\vec{p}}{16(c^2|\vec{p}|^2+m^2c^4)^3}[\alpha c^5(|\vec{E}(t)|^2|\vec{p}|^2 + \hbar\alpha^2c^4(\vec{E}(t)\cdot\vec{f}(t))^2]\right) = \exp\left(-\frac{\alpha\varepsilon(t)}{64mc^2} - \frac{\hbar\alpha^2cL^3}{16\pi(mc^2)^3}(\vec{E}(t)\cdot\vec{f}(t))^2\right)$$
$$\sim \exp\left(-\frac{\alpha\varepsilon(t)}{64mc^2}\right).$$

4. SCHWINGER'S RESULT

Here we consider the external uniform vector potential $\vec{f}(t) = (0, 0, \chi(t))$, where

$$\chi(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ cEt & \text{if} \quad 0 < t < T \\ cET & \text{if} \quad t > T, \end{cases}$$

and the spatial domain $[-L, L]^3$.

Then, $\forall t > T$, the probability that the vacuum state remains unchanged at time t is given by the Schwinger's formula

$$\exp\left(-\frac{TL^2E^2\alpha}{\pi^3\hbar}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\exp\left(-\frac{n\pi m^2c^4}{\hbar c|eE|}\right)\right).$$

Here we deduce this result using the relativistic tunneling effect (Eisenberg and Kälbermann, 1988; Marinov and Popov, 1977; Popov, 1972), i.e., using the WKB method in the complex plane.

If $0 < \tau < T$, the classical Hamiltonian is

$$H(\tau) = \pm\sqrt{c^2p_\perp^2 + c^2(p_3 + eE\tau)^2 + m^2c^4},$$

where $p_\perp = (p_1, p_2)$.

The dynamic equations are

$$\dot{x} = \frac{c^2 p_i}{H(\tau)}; \qquad i = 1, 2$$

$$\dot{x}_3 = \frac{c^2(p_3 + eE\tau)}{H(\tau)}$$

$$\dot{\vec{p}} = \vec{0}.$$

For a particle with negative kinetic energy and momentum $\vec{p}$, we have

$$x_3(\tau) = x_3(0) + \frac{1}{eE}\left(\sqrt{c^2|\vec{p}|^2 + m^2c^4} - \sqrt{c^2 p_\perp^2 + c^2(p_3 + eE\tau)^2 + m^2c^4}\right).$$

We note that, if $0 < \frac{-p_3}{eE} < T$, then $x_3(\frac{-p_3}{eE})$ is a classical turning point. Therefore, at $\frac{-p_3}{eE}$ there is a probability that the particle has positive kinetic energy, and then, if $\tau > \frac{-p_3}{eE}$, its dynamics would be

$$x_3(\tau) = x_3(0) + \frac{1}{eE}\left(\sqrt{c^2|\vec{p}|^2 + m^2c^3} - 2\sqrt{c^2 p_\perp^2 + m^2c^4}\right.$$
$$\left. + \sqrt{c^2 p_\perp^2 + c^2(p_3 + eE\tau)^2 + m^2c^4}\right).$$

The average number of produced pairs at time $t > T$ with momentum $(p_\perp, p_3)$, namely $\omega(p_\perp, p_3)$, is given in the adiabatic approach by the penetration factor

$$\omega(p_\perp, p_3) \sim \exp\left(-\frac{2}{\hbar}\int_{\tau-}^{\tau+} \sqrt{c^2 p_\perp^2 + m^2c^4 + c^2(p_3 + eE\tau)^2}\, d\tau\right),$$

where $\tau_\pm = \frac{-p_3 \pm \sqrt{p_\perp^2 + m^2c^2}}{eE}$, and $\tau_- < \tau < \tau_+$.

This penetration factor is obtained using the method of imaginary times (Popov, 1972), (note that, in a mathematical language this method is called "Over-Barrier Reflection" (Fedoryuk, 1993; Meyer, 1980)).

It is easy to verify that

$$\omega(p_\perp, p_3) \sim \exp\left(-\frac{\pi\left(c^2 p_\perp^2 + m^2c^4\right)}{\hbar c|eE|}\right).$$

Therefore, the probability that pairs with momentum $(p_\perp, p_3)$ are not produced is (Holstein, 1999; Nikishov, 1970; Parker, 1969)

$$\left(1 + \exp\left(-\frac{\pi\left(c^2 p_\perp^2 + m^2c^4\right)}{\hbar c|eE|}\right)\right)^{-1}.$$

In fact, let A be the probability that pairs with momentum $(p_\perp, p_3)$ are not produced, and let B be the relative pair production probability of a pair with momentum

$(p_\perp, p_3)$; then the absolute probability that n pairs with momentum $(p_\perp, p_3)$ are produced is AB^n (see (Nikishov, 1970, p. 350)). From the condition

$$A\sum_{n=0}^{\infty} B^n = 1,$$

we obtain $A = 1 - B$. Now, we use that the average number of produced pairs with momentum $(p_\perp, p_3)$ is

$$(1-B)\sum_{n=0}^{\infty} nB^n = \exp\left(-\frac{\pi\left(c^2 p_\perp^2 + m^2c^4\right)}{\hbar c|eE|}\right),$$

then, we deduce that

$$B = \frac{\exp\left(-\frac{\pi\left(c^2 p_\perp^2 + m^2c^4\right)}{\hbar c|eE|}\right)}{1+\exp\left(-\frac{\pi\left(c^2 p_\perp^2 + m^2c^4\right)}{\hbar_c|eE|}\right)},$$

consequently

$$A = \left(1+\exp\left(-\frac{\pi\left(c^2 p_\perp^2 + m^2c^4\right)}{\hbar c|eE|}\right)\right)^{-1}.$$

Once we have calculated the probability that pairs with momentum $(p_\perp, p_3)$ are not produced, we deduce that the probability that the vacuum state remains unchanged is

$$\prod\left(1+\exp\left(-\frac{\pi\left(c^2 p_\perp^2 + m^2x^4\right)}{\hbar c|eE|}\right)\right)^{-1}$$
$$= \exp\left(-\sum\log\left(1+\exp\left(-\frac{\pi\left(c^2 p_\perp^2 + m^2c^4\right)}{\hbar c|eE|}\right)\right)\right).$$

The sum is over all $p_\perp = \frac{\pi\bar{h}}{L}(k_1, k_2)$ with $(k_1, k_2) \in \mathbb{Z}^2$ and over p_3. But the time required by the particle to arrive at the turning point is $-\frac{p_3}{eE}$ if $0 < \frac{-p_3}{eE} < T$. Therefore, the particles that arrive at the turning point verify that p_3 is between 0 and eET. Therefore, since $\hbar$ is small, the sum is approximately an integral, and using the logarithm Taylor's series, we have

$$\exp\left(-\frac{TL^3|eE|}{(\pi\hbar)^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_{\mathbb{R}^2}\exp\left(-\frac{n\pi\left(2p_\perp^2 + m^2c^4\right)}{\hbar c|eE|}\right)dp_\perp\right)$$
$$= \exp\left(-\frac{TL^3E^3\alpha}{\pi^2\hbar}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\exp\left(-\frac{n\pi m^2c^4}{\hbar c|eE|}\right)\right).$$

Consequently, the Schwinger's formula gives the probability that the vacuum state remains unchanged for large times, i.e., for times such that the electric field is zero ($\chi(t) = cET$). If we compute the probability that the vacuum state remains unchanged for $0 < t < T$ we obtain the formula (1).

Remark 1. The probability that pairs with spin $\frac{1}{2}$ are not produced, using the Exclusion Principle, is

$$1 - \exp\left(-\frac{\pi\left(c^2 p_\perp^2 + m^2 c^4\right)}{\hbar c|eE|}\right).$$

Then, the probability that the vacuum remains unchanged is

$$\left(\prod 1 - \exp\left(-\frac{\pi\left(c^2 p_\perp^2 + m^2 c^4\right)}{\hbar c|eE|}\right)\right)^2,$$

since there are two different states for particles with spin $\frac{1}{2}$. Therefore, the final result is

$$\exp\left(-\frac{2TL^3E^2\alpha}{\pi^3\hbar}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left(-\frac{n\pi m^2c^4}{\hbar c|eE|}\right)\right).$$

Remark 2. The original method to derive the Schwinger's formula is founded in the definition of a relativistic invariant vacuum action, namely W (Greiner *et al.*, 1995; Itzykson and Zuber, 1980; Schwinger, 1951). This action has an imaginary part, then

$$\exp\left(-\frac{2}{\hbar} Im W\right),$$

"represents" the probability that the vacuum state remains unchanged.

There is a great analogy between the original method and the relativistic tunneling effect, because the penetration factor is

$$\exp\left(-\frac{2}{\hbar} Im S\right),$$

where S is the classical action computed along a simple closed curve, in the complex plane, containing the complex turning points $\frac{-p_3 \pm i\sqrt{p_\perp^2 + m^2c^2}}{eE}$ as interior points (Fedoryuk, 1993; Popov, 1972).

Remark 3. In the adiabatic approach, the pair production probability for large times involves always a factor that is exponentially small in $\hbar$, because in the adiabatic approach, the penetration factor (in our context the average number of produced pairs) is exponentially small in $\hbar$ (Bonet *et al.*, 1998; Fedoryuk, 1993; Meyer, 1980; Wasow, 1973).

5. THE STOCHASTIC PROCESS OF PAIR PRODUCTION

Here we consider the random variables $N(t) ='' $ net number of produced pairs at time t'', with

$$P(N(0) = n) = \begin{cases} 1 & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n \neq 0, \end{cases}$$

therefore

$$P(N(t) = n) = \sum_{\vec{k}_1,\ldots,\vec{k}_n} \frac{1}{n!} \left| (t)\langle 1^+_{\vec{k}_1} 1^-_{-\vec{k}_1} \ldots 1^+_{\vec{k}_n} 1^-_{-\vec{k}_n} \left| \mathcal{T}^t_\hbar \right| 0\rangle(0) \right|^2,$$

wherethe state $|1^+_{\vec{k}_1} 1^- \ldots 1^+_{\vec{k}_n} 1^-_{\vec{k}_n}\rangle(t)$ contains n particles whose momenta are $\vec{k}_1, \ldots, \vec{k}_n$ and n antiparticles whose momenta are $-\vec{k}_1, \ldots, -\vec{k}_n$.

Using the results obtained in Appendix A, it is easy to prove that if $\vec{f}(t) \in \mathcal{C}^\infty_0(0, \infty)$, then

$$P(N(t) = n) \sim \frac{1}{n!} \left(\frac{\alpha}{64mc^2} \varepsilon(t) \right)^n \exp\left(-\frac{\alpha}{64mc^2} \varepsilon(t) \right). \qquad (9)$$

Consequently, in the semiclassical approximation $N(t)$ is a stochastic Poisson process with an expected value $\frac{\alpha}{64mc^2}\varepsilon(t)$ (see (Schwinger, 1970)).

It is interesting to note that the process of the photon emission by a classical charged body, follows an exact stochastic Poisson process (see (Itzykson and Zuber, 1980; Schwinger, 1970).

APPENDIX A

The crucial part of the proof of Theorem 3.1 is the following lemma

Lemma 1. *The solution of the problem (8) is*

$$T^t_\hbar \phi^{0,0}_{\vec{k}}(0) = A_{\vec{k}}(t) \phi^{0,0}_{\vec{k}}(t) + \gamma_{\vec{k}}(t),$$

where

$$|A_{\vec{k}}(t)|^2 = 1 - \hbar^2 \frac{\epsilon^2_{\vec{k}}(t)}{16\epsilon^4_{\vec{k}}(t)} + h^4 J_{\vec{k}}(t),$$

with $|J_{\vec{k}}(t)| \leq K/\epsilon^4_{\vec{k}}$.

And $\gamma_{\vec{k}}(t)$ *satisfies*

$$\int_{\mathbb{R}_2} |\gamma_{\vec{k}}(t)|^2\, dQ_{\vec{k}}\, d\overline{Q}_{\vec{k}} \leq K \big/ \epsilon_{\vec{k}}^4 \quad \textit{and} \quad \gamma_{\vec{k}}(t) \perp \phi_{\vec{k}}^{0,0}(t),$$

where K *is a constant that is independent of* $\vec{k}$, $\hbar$, *and* t, *and* $\epsilon_{\vec{k}} = \sqrt{c^2|\frac{\pi\hbar\vec{k}}{L}|^2 + m^2c^4}$.

Proof of Lemma 1: First we will construct a semiclassical solution to problem (8). To find a semiclassical solution, we have to consider the functions $\phi_{\vec{k}}^{s,s}(t) = \frac{(\hat{a}_{\vec{k}}^+(t))^s(\hat{b}_{-\vec{k}}^+(t))^s}{s!}\phi_{\vec{k}}^{0,0}(t)$ with $s \in \mathbb{N}$.

We can write problem (8) in the form

$$\begin{cases} i\hbar\partial_t\phi = \hat{H}_{\vec{k}}(t)\phi \\ \phi(0) = \phi_{\vec{k}}^{0,0}(0), \end{cases} \tag{10}$$

where $\hat{H}_{\vec{k}}(t) = \epsilon_{\vec{k}}(t)(\hat{a}_{\vec{k}}^+(t)\hat{a}_{\vec{k}}(t) + \hat{b}_{-\vec{k}}^+(t)\hat{b}_{-\vec{k}}(t))$. If we expand the solution in power series of $\hbar$ thus, $T_\hbar^t\phi_{\vec{k}}^{0,0}(0) = \sum_{j,s\in\mathbb{N}} \hbar^{s+j} A_{s,\vec{k}}^j(t)$. Then, using the crucial result

$$\dot{\phi}_{\vec{k}}^{s,s}(t) = \frac{\dot{\epsilon}_{\vec{k}}(t)}{2\epsilon_{\vec{k}}(t)}\left(s\phi_{\vec{k}}^{s-1,s-1}(t) - (s+1)\phi_{\vec{k}}^{s+1,s+1}(t)\right),$$

we obtain, after having equalized the powers of $\hbar$, the system:

If $s = 0$

$$\dot{A}_{0,\vec{k}}^0 = 0; \quad \dot{A}_{0,\vec{k}}^j + \frac{\dot{\epsilon}_{\vec{k}}(t)}{2\epsilon_{\vec{k}}(t)} A_{1,\vec{k}}^{j-1} = 0, \quad \text{for} \quad j > 0.$$

If $s > 0$

$$-i\frac{\dot{\epsilon}_{\vec{k}}(t)}{2\epsilon_{\vec{k}}(t)} A_{s-1,\vec{k}}^0 - 2\epsilon_{\vec{k}}(t)A_{s,\vec{k}}^0 = 0.$$

$$i\dot{A}_{s,\vec{k}} - i\frac{\dot{\epsilon}_{\vec{k}}(t)}{2\epsilon_{\vec{k}}(t)} sA_{s-1\vec{k}}^1 - 2s\epsilon_{\vec{k}}(t)A_{s,\vec{k}}^1 = 0.$$

$$i\dot{A}_{s,\vec{k}}^{j-1} + i\frac{\dot{\epsilon}_{\vec{k}}(t)}{2\epsilon_{\vec{k}}(t)}\left((s+1)A_{s+1,\vec{k}}^{j-2} - sA_{s-1,\vec{k}}^j\right) - 2s\epsilon_{\vec{k}}(t)A_{s,\vec{k}}^j = 0, \quad \text{for} \quad j > 1.$$

We can obtain the solution to this system by recurrence. In fact,

$$A_{0,\vec{k}}^0(t) \equiv 1; \qquad A_{1,\vec{k}}^0(t) = -i\frac{\dot{\epsilon}_{\vec{k}}(t)}{4\epsilon_{\vec{k}}^2(t)}; \qquad A_{0,\vec{k}}^1(t) = \int_0^t i\frac{\dot{\epsilon}_{\vec{k}}^2(\tau)}{8\epsilon_{\vec{k}}^2(\tau)}\,d\tau.$$

$$A_{1,\vec{k}}^1(t) = \frac{1}{2\epsilon_{\vec{k}}(t)}\left(i\dot{A}_{1,\vec{k}}^0 - i\frac{\dot{\epsilon}_{\vec{k}}(t)}{2\epsilon_{\vec{k}}(t)}A_{0,\vec{k}}^1\right); \qquad A_{2,\vec{k}}^0(t) = -i\frac{\dot{\epsilon}_{\vec{k}}(t)}{4\epsilon_{\vec{k}}^2(t)}A_{1,\vec{k}}^0(t).$$

$$A_{0,\vec{k}}^2(t) = -\int_0^t \frac{\dot{\epsilon}_{\vec{k}}(\tau)}{2\epsilon_{\vec{k}}(\tau)}A_{1,\vec{k}}^1(\tau)\,d\tau$$

$$A^2_{1,\vec{k}}(t) = \frac{1}{2\epsilon_{\vec{k}}(t)}\left(i\dot{A}^1_{1,\vec{k}} + i\frac{\dot{\epsilon}_{\vec{k}}(t)}{2\epsilon_{\vec{k}}(t)}\left(2A^0_{2,\vec{k}} - A^2_{0,\vec{k}}\right)\right)$$

$$A^0_{3,\vec{k}}(t) = -i\frac{\dot{\epsilon}_{\vec{k}}(t)}{4\epsilon^2_{\vec{k}}(t)A^0_{2,\vec{k}}(t)}; \qquad A^3_{0,\vec{k}}(t) = -\int_0^t \frac{\dot{\epsilon}_{\vec{k}}(\tau)}{2\epsilon_{\vec{k}}(\tau)}A^2_{1,\vec{k}}(\tau)\,d\tau$$

$$A^1_{2,\vec{k}}(t) = \frac{1}{4\epsilon_{\vec{k}}(t)}\left(i\dot{A}^0_{2,\vec{k}} - i\frac{\dot{\epsilon}_{\vec{k}}(t)}{\epsilon_{\vec{k}}(t)}A^1_{1,\vec{k}}\right); \qquad \text{etc.} \qquad \square$$

With these solutions, and the relation $\epsilon^2_{\vec{k}} \leq C\epsilon^2_{\vec{k}}(t)$, where $C = 2(1 + \frac{e^2\|\vec{f}\|^2_\infty}{m^2c^4})$, we obtain

Lemma 2. *If $s, j \leq 4$ we have:*

$$\left|A^j_{s,\vec{k}}(t)\right| \leq \frac{\bar{C}}{\epsilon^{2s+j}_{\vec{k}}} \quad \textit{for} \quad s > 0; \qquad \left|A^j_{0,\vec{k}}(t)\right| \leq \frac{\bar{C}}{\epsilon^{2+j}_{\vec{k}}} \quad \textit{for} \quad j > 0$$

$$\left|\dot{A}^j_{s,\vec{k}}(t)\right| \leq \frac{g(t)}{\epsilon^{2s+j}_{\vec{k}}} \quad \textit{for} \quad s > 0; \qquad \left|\dot{A}^j_{0,\vec{k}}(t)\right| \leq \frac{g(t)}{\epsilon^{2+j}_{\vec{k}}} \quad \textit{for} \quad j > 0,$$

where $\bar{C}$ is a constant that is independent of $\vec{k}$, and $g(t) \in \mathcal{C}^\infty_0(0, \infty)$ is a function that is independent of $\vec{k}$. Now, we show that the function

$$\bar{\phi}_{\vec{k}}(t) = \sum_{\substack{s,j=0\\ s+j=4}} \hbar^{s+j} A^j_{s,\vec{k}}(t)\phi^{s,s}_{\vec{k}}(t),$$

is a semiclassical solution. In fact, using (Harthong, 1984; Maslov and Fedoriuk, 1981),

$$\sqrt{\int_{\mathbb{R}^2}\left|T^t_\hbar\phi^{0,0}_{\vec{k}}(0) - \bar{\phi}_{\vec{k}}(t)\right|^2 dQ_{\vec{k}}\,d\bar{Q}_{\vec{k}}} \leq \frac{1}{\hbar}\int_0^t \sqrt{|(i\hbar\partial_\tau - \hat{H}_{\vec{k}}(\tau))\bar{\phi}_{\vec{k}}(\tau)|^2\,dQ_{\vec{k}}}\,d\tau,$$

and Lemma 2, we obtain

$$\sqrt{\int_{\mathbb{R}^2}\left|T^t_\hbar\phi^{0,0}_{\vec{k}}(0) - \bar{\phi}_{\vec{k}}(t)\right|^2 dQ_{\vec{k}}\,d\bar{Q}_{\vec{k}}} \leq \frac{\hbar^4 G(t)}{\epsilon^4_{\vec{k}}},$$

where $G(t) \in \mathcal{L}^\infty(0, \infty) \cap \mathcal{C}^\infty(0, \infty)$ is independent of $\vec{k}$ and $\hbar$.

Therefore, $\overline{\phi}_{\vec{k}}(t)$ is a semiclassical solution (Hagedorn, 1980; Haro, 1998; Maslov and Fedoriuk, 1981), and we see that $T_{\hbar}^{t}\phi_{\vec{k}}^{0,0}(0)$ has the form

$$T_{\hbar}^{t}\phi_{\vec{k}}^{0,0}(0) = \overline{\phi}_{\vec{k}}(t) + \hbar^4 \sum_{s=0}^{4} F_{\vec{k},s}(t)\phi_{\vec{k}}^{s,s}(t) + \hbar^4 \beta_{\vec{k}}(t),$$

with $|F_{\vec{k},s}(t)| \leq \overline{K}/\epsilon_{\vec{k}}^4$, $\sqrt{\int_{\mathbb{R}^2} |\beta_{\vec{k}}(t)|^2 d\,Q_{\vec{k}} d\,\overline{Q}_{\vec{k}}} \leq \overline{K}/\epsilon_{\vec{k}}^4$, where $\overline{K}$ is a constant that is independent of $\vec{k}$ and $\hbar$, and $\beta_{\vec{k}}(t) \perp \phi_{\vec{k}}^{0,0}(t), \ldots, \phi_{\vec{k}}^{4,4}(t)$.

Now, we take

$$A_{\vec{k}}(t) = \sum_{j=0}^{4} \hbar^j A_{0,\vec{k}}^{j}(t) + \hbar^4 F_{\vec{k},o}(t)$$

$$\gamma_{\vec{k}}(t) = T_{\hbar}^{t}\phi_{\vec{k}}^{0,0}(0) - A_{\vec{k}}(t)\phi_{\vec{k}}^{0,0}(t).$$

Finally, since

$$\begin{aligned} A_{0,\vec{k}}^{2}(t) &= -\int_0^t \frac{\dot{\epsilon}_{\vec{k}}(\tau)}{4\epsilon_{\vec{k}}^2(\tau)}\left(i\dot{A}_{1,\vec{k}}^{0} - i\frac{\dot{\epsilon}_{\vec{k}}(\tau)}{2\epsilon_{\vec{k}}(\tau)}A_{0,\vec{k}}^{1}\right)d\tau \\ &= -\frac{\dot{\epsilon}_{\vec{k}}^2(t)}{32\epsilon_{\vec{k}}^4(t)} - \frac{1}{2}\left(\int_0^t \frac{\dot{\epsilon}_{\vec{k}}^2(\tau)}{8\epsilon_{\vec{k}}^3(\tau)}\,d\tau\right)^2, \end{aligned}$$

and $A_{0,\vec{k}}^{3}(t)$ is imaginary, we have

$$|A_{\vec{k}}(t)|^2 = 1 - \hbar^2 \frac{\dot{\epsilon}_{\vec{k}}^2(t)}{16\epsilon_{\vec{k}}^4(t)} + h^4 J_{\vec{k}}(t). \qquad \square$$

Proof of Theorem 3.1: Starting from the relation $|A_{\vec{k}}(t)|^2 = 1 - \hbar^2 \frac{\dot{\epsilon}_{\vec{k}}^2(t)}{16\epsilon_{\vec{k}}^4(t)} + h^4 J_{\vec{k}}(t)$, we have

$$\begin{aligned} P_{\hbar}(t) &= \prod_{\vec{k}\in\mathbb{Z}^3} |A_{\vec{k}}(t)|^2 = \prod_{\vec{k}\in\mathbb{Z}^3}\left(1 - \hbar^2 \frac{\dot{\epsilon}_{\vec{k}}^2(t)}{16\epsilon_{\vec{k}}^4(t)} + \hbar^4 J_{\vec{k}}(t)\right), \\ &= \exp\left(\sum_{\vec{k}\in\mathbb{Z}^3} \log\left(1 - \hbar^2 \frac{\dot{\epsilon}_{\vec{k}}^2(t)}{16\epsilon_{\vec{k}}^2(t)}\right) + O(\hbar)\right) \\ &= \exp\left(-\sum_{\vec{k}\in\mathbb{Z}^3} \hbar^2 \frac{\dot{\epsilon}_{\vec{k}}^2(t)}{16\epsilon_{\vec{k}}^4(t)} + O(\hbar)\right), \end{aligned}$$

where $O(\hbar)$ verifies $\lim_{\hbar\to 0} O(\hbar) = 0$, for fixed α.

If we use the Riemann's integral definition

$$\sum_{\vec{k}\in\mathbb{Z}^3} \frac{\hbar^2\dot{\epsilon}^2_{\vec{k}}(t)}{16\epsilon^4_{\vec{k}}(t)} = \frac{L^2\alpha c^5}{16\pi^3}\int_{\mathbb{R}^3} \frac{(\vec{E}(t)\cdot\vec{p})^2}{(c^2|\vec{p}|^2+m^2c^4)^3}\,d\vec{p} + O(\hbar),$$

then we obtain

$$P_\hbar(t) = \exp\left(-\frac{L^3\alpha c^5}{16\pi^3}\int_{\mathbb{R}^3} \frac{(\vec{E}(t)\cdot\vec{p})^2}{(c^2|\vec{p}|^2+m^2c^4)^3}\,d\vec{p} + O(\hbar)\right).$$

Now it is easy to prove that

$$p_\hbar(t) = \exp\left(-\frac{\alpha}{64}\frac{\varepsilon(t)}{mc^2} + O(\hbar)\right) \sim \exp\left(-\frac{\alpha}{64}\frac{\varepsilon(t)}{mc^2}\right). \qquad \square$$

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